Introduction to Structured Derivations

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Four Ferries Publishing

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Preface

Structured derivations is a method that supports the construction, presentation and understanding of mathematical arguments. The method works equally well for mathematical proofs and algebraic and arithmetic calculations as for geometric constructions and general problem solving, and it is useful whenever the presentation of a solution requires several consecutive steps. It has been used at different levels of mathematics, from lower secondary school to university level and research. The method is based on a fixed form to present mathematical arguments and the use of simple logical notation in the arguments. The fixed form makes it easier to to understand proofs and calculations and to check that they are correct.

The aim of this guide is to show how structured derivations can be used in upper secondary school level mathematics education. The method is described with examples that step by step introduce new features and concepts. This guide is an expanded English version of a previous manual, *Matematiikkaa logiikan avulla: Johdatus rakenteisiin päättelyketjuihin* (Ralph-Johan Back, TUCS Lecture Notes 10, 2008).

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CHAPTER 1

Introduction

This is a tutorial for *structured derivations.* Structured derivations are written in a fixed format that shows the overall structure of the mathematical argument and how the different parts of the argument are connected to each other. The presentation format demands that each step of a derivation is explicitly justified, i.e., a reason must be given for why the step is mathematically correct. Basic logical symbols are used explicitly in derivations, to speed up the argumentation and show the logical structure of a proposition. This is similar to how arithmetic and algebraic notation is used in algebraic manipulations. Structured derivations provide a unified presentation format for different kinds of mathematical reasoning: proofs, arithmetic and algebraic calculations, geometric proofs, solutions to verbal problems, and so on. The method is suitable for all levels of mathematics, from lower secondary school to university teaching and research.

The aim of this guide is to provide an overview of structured derivations using examples taken from high school math. Features of the method are introduced one at a time, and are illustrated with a number of examples. Reading the text does not require any other background information than traditional high school mathematics.

Chapter 2

Structured Tasks

A structured task combines a mathematical problem with its solution into a single presentation with a fixed format. A structured task can be a mathematical theorem and its proof, an algebraic task and its solution, a verbal task and its solution and so on. We show in this chapter how to present different mathematical problems and their solutions as structured tasks.

2.1 Structured Calculation

First we present a simple special case of a structured task, a *structured calculation*. An algebraic calculation is usually described as a chain of equalities,

$$t_0 = t_1 = ... = t_n.$$

This chain corresponds to the propositions

$$t_0 = t_1 \text{ and } t_1 = t_2 \text{ and } \ldots t_{n-1} = t_n$$

From this we can conclude that

$$t_0 = t_n,$$

since equality is transitive. The following example shows how we write such a calculation as a *structured task*.

Example 1. Show that the formula for the difference of two squares,

$$(a + b)(a - b) = a^2 - b^2$$

is true.

We prove this proposition with a structured calculation. Each expression is then written on a separate line. The justification for an equality is written on a line of its own, after the equality sign and between the expressions that form the equality.

The proof in the form of a structured calculation looks as follows:

$$
\begin{array}{ll}
\bullet & (a-b)\,(a+b) \\
= & \{\text{the rule for multiplying polynomials}\} \\
 & a^2 + ab - ba - b^2 \\
= & \{\text{the expressions } ab \text{ and } -ba \text{ cancel out}\} \\
 & a^2 - b^2 \\
\square &
\end{array}
$$

The proof is a chain of three algebraic expressions, which we can combine with equalities:

$$(a-b)\,(a+b) \;=\; a^2 + ab - ab - b^2 \;=\; a^2 - b^2$$

This proves the proposition, since equality is a transitive relation. We justify the equalities by the properties of polynomials. ■

Traditionally, we would write the calculation in the following way:

$$
\begin{array}{lll}
(a-b)\,(a+b) & = a^2 + ab - ba - b^2 & \text{(by the rules for multiplying polynomials)} \\
 & = a^2 - b^2 & \text{(the expressions } ab \text{ and } -ba \text{ cancel out)}
\end{array}
$$

In this form, both the formula and the justification must fit on the same line. This means that the justification must be short, which often cryptic, and it is easy to omit it from the proof. The basic idea of a mathematical proof is then lost, i.e., to convince the reader that each step is correct and that the steps combined lead to the desired conclusion. If justifications are omitted, the readers must find them themselves. This makes it harder to read and understand a proof. At the same time, the risk of misunderstandings and errors increase, which in turn will make it more difficult to understand subsequent argumentations.

From a pedagogical point of view, it is particularly important that all the justifications are written out explicitly. The teacher is then able to check that a student has a real understanding of the theory needed, and has applied it correctly. It is also easier for the students to check his or her own calculations, by going through each step separately, checking the justification for the step, and that the step has been carried out correctly.

The whole line is reserved for justifications in a structured calculation, which means tat there is enough space for proper justifications. More lines can be used if needed. In a way, the format forces one to give a justification for each step, since an omitted justification stands out as an empty parenthesis.

A structured calculation has the following general format:

$\bullet$	t_0
$\sim_1$	{explanation for why proposition $t_0 \sim_1 t_1$ is true}
	t_1
$\sim_2$	{explanation for why propsition $t_1 \sim_2 t_2$is true}
	t_2
$\vdots$	
	t_{n-1}
$\sim_n$	{explanation for why proposition $t_{n-1} \sim_n t_n$ is true}
	t_n
$\square$	

All relations $\sim_i$ are equalities in the previous example, but in general, this need not be the case. Each proposition in the calculation is justified by an argument written between curly brackets after the relation symbol. Each expression and each justification is written on a line of its own. In this way, we get more space for longer expressions and justifications. If needed, we can use more than one line for an expression or justification.

The calculation is written in two columns: the first one contains the relation symbols (in the example "=") and the special characters (here the "•" that begins the derivation and the "□" that ends it), the second one contains the expressions and the justifications.

We can use any binary relation $\sim$ between the expressions. The most common ones are $\equiv$ or $\Leftrightarrow$ (equivalence), $\Rightarrow$ (implication) and $\Leftarrow$ (reverse implication) for logical propositions, together with equality $=$ and order relations $<, >, \leq, \geq$ for arithmetic and algebraic propositions. Available relations are not limited to just these, other relations can be used as well. We often choose transitive relations (all the relations mentioned above are transitive), but we can also use other binary relations between the expressions. We can also use different binary relations in the same derivation. For instance, equality can be combined with any relation: if $a \sim b$ holds and $b = c$, then proposition $a \sim c$ also holds.

We use the the notation$\equiv$ for equivalence between logical expressions in our examples. The relation $p \equiv p'$, where p and p' are logical propositions, states that p and p' are equally true, i.e., either both are true or both are false. Another common notation for equivalence is $\Leftrightarrow$, but we think that the notation $\equiv$ better illustrates that we are talking about equality between truth values. The equivalence relation is transitive.

An example of a non-transitive relation is inequality: $a \neq b$ and $b \neq c$ does not necessarily mean that $a \neq c$. A simple example to the contrary is $0 \neq 1$ and $1 \neq 0$, which are both true, but $0 \neq 0$ is not true. Therefore, using several consecutive inequalities in a calculation would generally do not lead to a desired result.

2.2 Structured Task

We often want to state explicitly what task we should solve and what assumptions we may assume when solving the task. We then use a general *structured task.* The following example illustrates this format.

Example 2. Show that if a,b and c are non-negative numbers, then it holds that

$$(1+a)\,(1+b)\,(1+c) \geq 1+a+b+c$$

We formulate the task, the assumptions and the calculation that solves the task as a structured task.

$\bullet$	Show that $(1+a)\,(1+b)\,(1+c) \geq 1+a+b+c$, when
$-$	$a,b,c \geq 0$
$\Vdash$	$(1+a)\,(1+b)\,(1+c)$
$=$	{multiply the expressions $(1+b)$ and $(1+c)$ by each other}
	$(1+a)\,(1+b+c+bc)$
$=$	{multiply the expressions $(1+a)$ and $(1+b+c+bc)$ by each other}
	$1+b+c+bc+a+ab+ac+abc$
$\geq$	{subtract the non-negative expression $ab+ac+bc+abc$ from the expression}
	$1+a+b+c$
$\square$	

The task that we want to solve is,

Show that $(1+a)\,(1+b)\,(1+c) \geq 1+a+b+c$

The task is written directly after the task sign $\bullet$, in the second column. The following lines list the assumptions that we are allowed to make. In this example there is only one assumption,

$a,b,c \geq 0$

The proof itself is a structured calculation, which begins after the $\Vdash$-sign, and ends at the $\Box$-sign, just as before. ■

A structured calculation is a special case of a structured task, where the task and the assumptions are omitted. This shorter form is suitable in situations where it is clear from the context what we want to prove and which assumptions hold. A structured task is used when we want to spell out exactly the task to be solved and the assumptions. In practice both formats are used in parallel.

The basic format of a structured task is as follows:

•	Task
-	assumption_1
$\vdots$	
-	assumption_m
$\Vdash$	{explanation for why the calculation solves the task under the given assumptions}
	t_0
$\sim_1$	{justification for the proposition $t_0 \sim_1 t_1$}
	t_1
$\sim_2$	{justification for the proposition $t_1 \sim_2 t_2$}
	t_2
$\vdots$	
	t_{n-1}
$\sim_n$	{justification for the proposition $t_{n-1} \sim_n t_n$}
	t_n
$\Box$	

An assumption is marked by a ”−”. We can also mark the assumptions with letters in parentheses (such as (a), (b) , ...), if we need to reference individual assumptions in the calculation. The proof that follows the assumptions begins with the $\Vdash$-symbol (read ”prove that”). The proof ends by the symbol $\Box$ (read ”which was to be proven”). We may add a justification for why the calculation proves the proposition under the given assumptions, after the proof symbol. We use two columns as before. We write special symbols like "$\bullet$", "$\Vdash$" ,"$-$" and relation symbols in the first column and the task, assumptions, expressions and justifications in the second column.

We will continue below by expanding this basic format with new functionality, such as nested tasks, facts and definitions. However, the basic format above works well for solving many simpler mathematical problems.

2.3 Nested Tasks

The task format above is sufficient when the justification for each step is relatively simple and can be condensed into a few lines. When the justifications get more complicated, we will need *nested tasks*. Below is an example of how to use nested tasks in a derivation.

Example 3. Show that $m^2 - n^2 \geq 3$, when m and n are positive integers and $m > n$. The following two basic rules of arithmetics are used in this derivation:

$$\begin{array}{ll} a + b \leq a + b', \text{ when } b \leq b' & \text{(addition is monotonic)} \\ ab \leq ab', \text{ when } a \geq 0 \text{ and } b \leq b' & \text{(multiplication is monotonic)} \end{array}$$

We prove the proposition in the following way.

$\bullet$	Show that $m^2 - n^2 \geq 3$, when
$-$	m and n are positive integers and
$-$	$m > n$
$\Vdash$	$m^2 - n^2$
$=$	{the difference of two squares}
	$(m - n)(m + n)$
$\geq$	{multiplication is monotonic, because $m - n \geq 0$ and $m + n \geq 3$ follow from the assumptions}
	$(m - n) \cdot 3$
$\geq$	{multiplication is monotonic, because $m - n \geq 1$ and $3 \geq 0$ follow from the assumptions}
	$1 \cdot 3$
$=$	{calculate}
	3
$\square$	

$\blacksquare$

The second step of the derivation is somewhat complicated. It refers to the rule for monotonicity of multiplication. However, the rule is only applicable when the conditions $m - n \geq 0$ and $m + n \geq 3$ are satisfied. We can reason as follows:

- $m > n$ according to the assumption, so $m - n > 0$ (and hence also $m - n \geq 0$),
- $n > 0$ according to the assumption, so $n \geq 1$, and
- $m > n \geq 1$ according to the assumption, so $m \geq 2$ and hence $m + n \geq 3$.

We can write these justifications as simple (logical) calculations in the following way:

•	$m > n$	•	$m > n$ and $n > 0$
$\equiv$	{arithmetics}	$\equiv$	{arithmetics}
	$m - n > 0$		$m \geq n+1$ and $n \geq 1$
$\Rightarrow$	{arithmetics}	$\Rightarrow$	{arithmetics}
	$m - n \geq 0$		$m \geq 2$ and $n \geq 1$
$\square$		$\Rightarrow$	{arithmetics}
			$m + n \geq 3$
		$\square$	

Instead of writing these additional justifications separately in the proof, we can write them directly in the proof as *nested tasks*. Nested tasks are written directly after a justification, but indented one level to the right. This shows that they are part of the justification for the step. Assumptions made in a task will also be valid in subsequent nested tasks. Later we will show that nested tasks can also include assumptions of their own.

Example 4. We rewrite the previous proof using nested tasks.

$\bullet$ Show that $m^2 - n^2 \geq 3$, when

$-$ m and n are positive integers, and

$-$ $m > n$

$\Vdash$ $m^2 - n^2$

$=$ {the difference of two squares}

$(m - n)\,(m + n)$

$\geq$ {multiplication is monotonic, as assumptions $m - n \geq 0$ and $m + n \geq 3$ hold}

- $\bullet$ $m > n$
- $\equiv$ {arithmetics}
- $m - n > 0$
- $\Rightarrow$ {arithmetics}
- $m - n \geq 0$
- $\square$
- $\bullet$ $m > n$ and $n > 0$
- $\equiv$ {arithmetics}
- $m \geq n + 1$ and $n \geq 1$
- $\Rightarrow$ {arithmetics}
- $m \geq 2$ and $n \geq 1$
- $\Rightarrow$ {arithmetics}
- $m + n \geq 3$
- $\square$

$\ldots$ $(m - n) \cdot 3$

$\geq$ {multiplication is monotonic, as assumptions $m - n \geq 1$ and $3 \geq 0$ hold}

$1 \cdot 3$

$=$ {calculate}

3

$\square$ $\blacksquare$

The format for a simple justification for the proof step $t \sim t'$ is shown below on the left. The format for a justification with nested derivations is shown below on the right.

$$\begin{array}{ll} & t \\ \sim & \{\text{justification}\} \\ & t' \end{array}$$

$$\begin{array}{ll} & t \\ \sim & \{\text{justification}\} \\ & t\text{ask}_1 \\ & \vdots \\ & task_n \\ \ldots & t' \end{array}$$

A justification can thus be supported by n different nested tasks: $task_1, \ldots,\ task_n$, $n \geq 0$. We indent these so that they begin at the following column. This makes nested tasks easy to distinguish from the main task. The justification explains why $t\ R\ t'$ is true, when we assume that each nested task is true. Since nested tasks can sometimes be quite long, we may mark the second expression (in this case t') with the symbol "..." in the first column. This makes it easier to see where the nested tasks end and the main task continues.

2.4 Focusing on Partial Expressions

Many calculations require us to work with long and complicated expressions. Nested calculations can then be used to focus on a part of an expression, modifying it without having to rewrite the entire expression in each step.

Example 5. Simplify the expression $\sqrt{7+2\sqrt{11}}+\sqrt{7-2\sqrt{11}}$.

$\bullet$ Simplify the expression $\sqrt{7+2\sqrt{11}}+\sqrt{7-2\sqrt{11}}$

$\Vdash$ $\sqrt{7+2\sqrt{11}}+\sqrt{7-2\sqrt{11}}$

$=$ {square the expression, simplify it and take the square root of the simplified expression}

> $\bullet$ $(\sqrt{7+2\sqrt{11}}+\sqrt{7-2\sqrt{11}})^2$
>
> $=$ {the square of a binomial}
>
> $7+2\sqrt{11}+2\cdot\sqrt{7+2\sqrt{11}}\cdot\sqrt{7-2\sqrt{11}}+7-2\sqrt{11}$
>
> $=$ {focus on the partial expression $2\cdot\sqrt{7+2\sqrt{11}}\cdot\sqrt{7-2\sqrt{11}}$}
>
> > $\bullet$ $2\cdot\sqrt{7+2\sqrt{11}}\cdot\sqrt{7-2\sqrt{11}}$
> >
> > $=$ {write under the same radical sign}
> >
> > $2\cdot\sqrt{(7+2\sqrt{11})\cdot(7-2\sqrt{11})}$
> >
> > $=$ {the difference of two squares}
> >
> > $2\sqrt{49-4\cdot 11}$
> >
> > $=$ {simplify}
> >
> > $2\sqrt{5}$
> >
> > $\square$
>
> $\ldots$ $7+2\sqrt{11}+2\sqrt{5}+7-2\sqrt{11}$
>
> $=$ {simplify}
>
> $14+2\sqrt{5}$
>
> $\square$

$\ldots$ $\sqrt{14+2\sqrt{5}}$

$\square$ $\blacksquare$

The example uses a nested calculation, which in turn contains another nested calculation. The original problem was to simplify a radical expression. We first simplify the square of the expression in a nested calculation. The square root of the simplified expression is then the solution to the original problem. The inner nested calculation focuses on only a part of an expression, so that is is easier to see what we are working on. We also make fewer mistake when we do not have to copy complicated expressions from one line to the next.

Writing and reading this type of proof becomes easier when we use a computer and an editor that support indentation and that can display and hide nested derivations (an *outlining editor*). We can then decide on the level of detail we use when studying the proof. Hiding nested derivations gives a better overview of the proof, while showing them gives a more detailed picture of the proof.

2.5 Word Problems

A word problem first describes a situation. The task is then to prove or calculate something related to this situation. We show here how to solve word problems with tasks. We have here indicated the assumptions by letters, so that it is easier to refer to them in the calculation. Our first word problem is from mechanics.

Example 10. Since the year 1960 the travel time of the fastest train connection between the cities of Helsinki and Lappeenranta has decreased by 37%. Calculate by how many percentages the average speed has increased. Assume that the length of the railroad has not changed.

We first rewrite the task by adding notations used in the proof. The rewritten task is as follows: After the year 1960 the travel time t' of the fastest train connection between Helsinki and Lappeenranta has shortened by 37% compared to the original travel time t. Calculate how by how many percent p the mean speed v' has increased compared to the original mean speed v. We assume that the length s of the track has not changed.

• Calculate the percentage of change p in the speed, when

$-$ $t' = 0.63 \cdot t$

$\Vdash$ p

$=$ {the definition of percentage of change}

$$\frac{v' - v}{v}$$

$=$ {physics: the definition of mean speed is $v = \frac{s}{t}$, where s is the distance travelled and t is the travel time}

$$\frac{\frac{s}{t'} - \frac{s}{t}}{\frac{s}{t}}$$

$=$ {simplify}

$$\frac{\frac{s}{t'}}{\frac{s}{t}} - 1$$

$=$ {simplify the fractions}

$$\frac{s \cdot t}{s \cdot t'} - 1$$

$=$ {simplify, assumptions}

$$\frac{1}{0.63} - 1$$

$\approx$ {calculate an approximate value}

0.59

$=$ {convert to percent: $x\% = \frac{x}{100}$}

59%

$\square$

Answer: the mean speed has increased by 59%. ■

2.6 Questions and Answers

A mathematical task is often to find a value x that satisfies some given conditions, or to find every value x that satisfy the given conditions. Solving an equation is an example of this common type of task. Often its is not stated explicitly with respect to which variable one should solve an equation, since this is obvious (the variable x), but when the equation has many variables and constants, then it may be ambiguous which variable value we need to find.

We can describe a task more precisely by stating explicitly the variable that we are interested in, and what conditions it should satisfy. We can then give the answer explicitly after the $\square$- symbol. The following word problem shows how this works.

Example 14 The width of a board is 95 mm and its length is 1,6 m. It is sawn into pieces of the same length, which are placed next to each other so that they form a board in the shape of a square. What is the maximum length of the side of the square?

Let n be the number of pieces, $n \in \mathbb{Z}_+$, and let x be the length of the pieces. Then we know that $nx \leq 1600$, where the unit of length is millimeter. We also know that $x = 95n$, since the pieces form a square. We describe the situation in *2.1* (notice that one piece may be left over) .

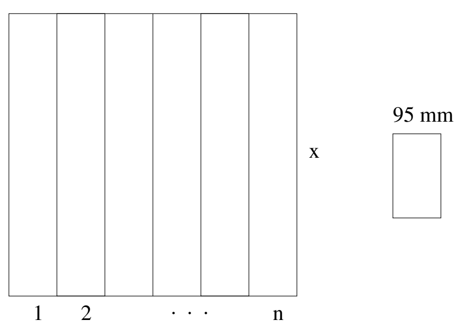


Figure 2.1: The square and the excess piece.

Now we want to maximize the value of x.

- Find $n \in \mathbb{Z}_+$ that maximizes the value $x = 95n$ and satisfies the condition $nx \leq 1600$

$\Vdash$ n maximizes the value $x = 95n$, and $nx \leq 1600$ and $n \in \mathbb{Z}_+$

$\equiv$ {simplify the condition}

  - Simplify $nx \leq 1600$, when

  \- $x = 95n$

  $\Vdash$ $nx \leq 1600$

  $\equiv$ {use the assumption $x = 95n$}

  $n \cdot 95n \leq 1600$

  $\equiv$ {simplify}

  $n^2 \leq 1600/95$

  $\equiv$ {solve by assuming that $n \in \mathbb{Z}_+$; $16 \leq \frac{1600}{95} < 25$}

  $n \leq 4$

  $\square$

... n maximizes the value $x = 95n$, and $n \leq 4$ and $n \in \mathbb{Z}_+$

$\equiv$ {x is a strictly increasing function of n}

$n = 4$

$\square$ $n = 4$

In other words, the largest possible board is $4 \cdot 95 = 380$ (mm). ■

Chapter **3**

Solving a Problem Step by Step

In the previous chapter we showed how structured tasks are applied to solve simple mathematical tasks. But when the solution becomes longer and more complicated, we need methods that allow us to split the solution into smaller parts. Furthermore we need a clear strategy for how to solve a task.

In mathematics there are three basic strategies for solving more complex problems step by step, algebraic calculation, forward proof and backward proof. We have shown in the previous chapter how to solve problems by calculation. In a forward proof we proceed step by step so that we list a sequence of facts that follow from the assumptions and previous facts and that help with the understanding and solving of the problem. We continue until we have collected so many facts that we can solve the problem directly, by calculating or by another fact. Backward proofs start from the original problem and tries to reduce this to as set of simpler problems, so that solving them also gives the solution to the original problem. We can solve simpler problems directly, for instance by calculating or then we can reduce them to even simpler problems, etc.

Structured derivations enable all these proof strategies at the same time, in different parts of a solution. In this chapter we show how to use forward and backward proofs in structured tasks.

3.1 Forward Proofs

When we solve a task we often have a situation where we cannot proceed to directly calculate the desired result. First we need to prepare for the calculation by listing a few *facts* that follow directly from the assumptions and that we need for the calculations or to prove other facts. We write facts in the following way:

$+$ {justification}

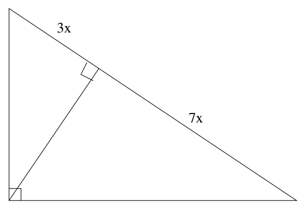
proposition

We mark facts with a ”+”-sign, contrary to assumptions, which we mark by a ”−”-sign. Facts should always include a justification, which tells how the fact follows from the assumptions and earlier facts. The justification is written before the observation. The justification can be simple or it can include nested tasks. Facts can be numbered, in which case the numbers are written in square brackets (e.g. “[1]”, “[2]” etc.).

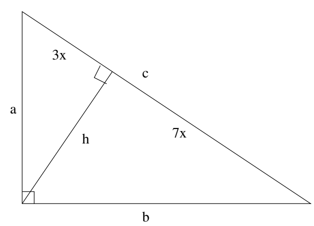
First we will show by an example how we can use observations in a derivation from geometry.

Example 11 The height to the hypotenuse in a right triangle divides the hypotenuse at a ratio of 3 : 7. Find the ratio between the legs of the triangle.

We start by drawing a figure, which describes the problem:



In the next figure we have named the legs (a and b), the hypotenuse (c) and the height (h).



We use this notation in the task. First we write some simple facts that follow directly from the figure. Based on these, we can prove a few other facts. Finally, we calculate the ratio between the length of the legs using these facts.

- Find the ratio $\frac{a}{b}$

[1] {from the figure}

$c = 10x$

[2] {the figure and the Pythagorean theorem}

$h^2 + 9x^2 = a^2$

[3] {the figure and the Pythagorean theorem}

$h^2 + 49x^2 = b^2$

[4] $\{$the figure and the Pythagorean theorem$\}$

$a^2 + b^2 = 100x^2$

[5] $\{$eliminate $h\}$

$\bullet$ [2] $\wedge$ [3]

$\equiv$ $\{$observation [2] and [3]$\}$

$h^2 + 9x^2 = a^2 \wedge h^2 + 49x^2 = b^2$

$\Rightarrow$ $\{$subtract the first equation from the second one and simplify$\}$

$b^2 - a^2 = 40x^2$

$\square$

$\ldots$ $b^2 - a^2 = 40x^2$

[6] $\{$Find $b^2\}$

$\bullet$ [4] $\wedge$ [5]

$\equiv$ $\{$observation [4] and [5]$\}$

$a^2 + b^2 = 100x^2 \wedge b^2 - a^2 = 40x^2$

$\Rightarrow$ $\{$add together the equations$\}$

$2b^2 = 140x^2$

$\equiv$ $\{$divide both sides by 2$\}$

$b^2 = 70x^2$

$\square$

$\ldots$ $b^2 = 70x^2$

[7] $\{$Find $a^2\}$

$\bullet$ [4] $\wedge$ [6]

$\equiv$ $\{$observation [4] och [6]$\}$

$a^2 + b^2 = 100x^2 \wedge b^2 = 70x^2$

$\Rightarrow$ $\{$substitute the second equation into the first one$\}$

$a^2 + 70x^2 = 100x^2$

$\equiv$ $\{$solve $a^2\}$

$a^2 = 30x^2$

$\square$

$\ldots$ $a^2 = 30x^2$

$\Vdash$ $\frac{a}{b}$

$=$ {the definition of square root, a and b are positive numbers}

$\sqrt{\frac{a^2}{b^2}}$

$=$ {observation [6] and [7]}

$\sqrt{\frac{30x^2}{70x^2}}$

$=$ {reduce and simplify}

$\frac{\sqrt{3}}{\sqrt{7}}$

□ ■

The following example from analytic geometry gives another example of how to use facts in derivations. In this example we identify the assumptions by letters and the facts by numbers.

Example 12. Find the point on the parabola $y = x^2 - 2x - 3$, where the directed angle of the tangent is $45°$.

We can rephrase the problem in the following way: Find the point (x_0, y_0) on the parabola $y = x^2 - 2x - 3$ where the directed angle α of the tangent is $45°$.

- Find the point (x_0, y_0), when

(a) $y = f(x) = x^2 - 2x - 3$ for every $x \in \mathbb{R}$, and

(b) the directed angle of the tangent at the point (x_0, y_0) is $\alpha = 45°$

[1] {find the derivative at the point x_0}

- the tangent to the parabola at the point (x_0, y_0) has the directed angle $45°$

  $\equiv$ {the slope k is given by the directed angle α with he formula $k = \tan\alpha$}

  the slope of the tangent at the point (x_0, y_0) is $\tan 45°$

  $\equiv$ {$\tan 45° = 1$}

  the slope of the tangent at the point (x_0, y_0) is 1

  $\equiv$ {the derivative of the function gives the slope}

  $f'(x_0) = 1$

  □

... $f'(x_0) = 1$

[2] {find the value of the variable x_0, observation [1]}

- $f'(x_0) = 1$

$$
\begin{array}{ll}
\equiv & \{\text{assumption } (a), \text{ calculate the derivative}\} \\
 & 2x_0 - 2 = 1 \\
\equiv & \{\text{solve } x\} \\
 & x_0 = \frac{3}{2} \\
\square &
\end{array}
$$

$$
\begin{array}{ll}
\ldots & x_0 = \frac{3}{2} \\
\Vdash & (x_0, y_0) \\
= & \{\text{observation [2]}\} \\
\ldots & \left(\frac{3}{2}, y_0\right) \\
= & \{\text{find the value of } y_0 \text{ using assumption } (a)\} \\
 & \left(\frac{3}{2}, \left(\frac{3}{2}\right)^2 - 2 \cdot \left(\frac{3}{2}\right) - 3\right) \\
= & \{\text{calculate}\} \\
 & \left(\frac{3}{2}, -\frac{15}{4}\right) \\
\square &
\end{array}
$$

Thus the point we are looking for is $(x_0, y_0) = (\frac{3}{2}, -\frac{15}{4})$. ■

The following is a general form of a structured task with facts.

•	Task
-	assumption$_1$
⋮	
-	assumption$_m$
+	{justification of the fact}
	fact$_1$
⋮	
+	{justification of the fact}
	fact$_m$
$\Vdash$	{justification for why the derivation is a solution to the task with the given assumptions and facts}
	t_0
$\sim_1$	{justification of the proposition $t_0 \sim_1 t_1$}
	t_1
$\sim_2$	{justification of the proposition $t_1 \sim_2 t_2$}
	t_2
⋮	
	t_{n-1}
$\sim_n$	{justification of the proposition $t_{n-1} \sim_n t_n$}
	t_n
□	

3.2 Backward Proofs

Backward proofs do not require new mechanisms, we can do them using nested tasks. We present two examples where the given task is solved by reducing the original task to more manageable nested tasks. We do not need a calculation at the task level, so we can omit this.

The first example is a proof by cases. In this type of proof we first identify every possible case, then we show that our proposition is true in each case. It is important that the cases that we treat cover every possibility.

Example 16. Prove that the inequality $|x + 1| > 1$ is true outside the interval $[-2,0]$.

• Show that $|x+1| > 1$, when

(a) $x > 0 \vee x < -2$

$\Vdash$ {proof by cases, according to assumption (a), checking the cases $x > 0$ and $x < -2$ separately is sufficient}

  • Show that $|x+1| > 1$, when

  $-$ $x > 0$

  $\Vdash$ $|x+1|$

  $=$ {the definition of absolute value, assumption $x > 0$}

  $x+1$

  $>$ {assumption}

  $0+1$

  $=$ {calculate}

  1

  $\square$

  • Show that $|x+1| > 1$, when

  $-$ $x < -2$

  $\Vdash$ $|x+1|$

  $=$ {the definition of absolute value, assumption $x < -2$}

  $-(x+1)$

  $=$ {simplify}

  $-x-1$

  $>$ {assumption $x < -2$ i.e. $-x > 2$}

  $2-1$

  $=$ {calculate}

  1

  $\square$

$\square$ $\blacksquare$

A proof by induction is another example where reduction is a good way to treat a problem. In a proof by induction we treat two cases: the base case and the inductive step. If we can prove both steps, then the induction hypothesis is true.

Example 14. Prove that

$$0 + 1 + \ldots + n = \frac{n\,(n+1)}{2}$$

for each $n \in \mathbb{N}$, using induction. We prove this as follows:

$\bullet$ Show that $0 + 1 + \ldots + n = \frac{n(n+1)}{2}$ for each $n \in \mathbb{N}$.

$\Vdash$ {proof by induction}

- $\bullet$ Base case: show that $0 + 1 + \ldots + n = \frac{n(n+1)}{2}$, when
- $-$ $n = 0$
- $\Vdash$ $0 + 1 + \ldots + n = \frac{n(n+1)}{2}$
- $\equiv$ {substitute assumption $n = 0$}
- $0 = \frac{0(0+1)}{2}$
- $\equiv$ {multiply by zero}
- T
- $\square$

- $\bullet$ Inductive step: show that $0 + 1 + \ldots + n' = \frac{n'(n'+1)}{2}$, when
- $-$ $n' = n + 1$ and
- $-$ $0 + 1 + \ldots + n = \frac{n(n+1)}{2}$
- $\Vdash$ $0 + 1 + \ldots + n'$
- $=$ {assumption}
- $0 + 1 + \ldots + n + (n + 1)$
- $=$ {induction hypothesis}
- $\frac{n(n+1)}{2} + (n + 1)$
- $=$ {find a common denominator}
- $\frac{n(n+1)+2(n+1)}{2}$
- $=$ {the distributive law, the commutative law}
- $\frac{(n+1)(n+2)}{2}$
- $=$ {assumption $n' = n + 1$}
- $\frac{n'(n'+1)}{2}$
- $\square$

$\square$ $\blacksquare$

Chapter **4**

Structured Derivations

A structured derivation is a sequence of derivation steps, where each step is either

- an assumption,
- a declaration,
- a fact,
- a definition, or
- a task.

A structured derivation is the traditional way in which a mathematician work on a problem. First, they try to formulate the problem in mathematical form. When they do this, they realize the need for making the assumptions more precise, and perhaps also the need for some additional assumptions. Similarly,they might need to introduce some new concepts in order to express the problem better. After this, they focus on solving the problem. When the original problem has been solved, they may see that it leads to other interesting problems, which can be studied the same context. These require additional assumptions and new definitions and lead to new problems, etc. Mathematical reasoning develops a bit like a novel, with a clear plot and a climax. The difference is that each step must be proven to be correct, since a single error can ruin the entire structure.

A structured derivation gives more freedom for expressing and solving a problem than using a single structured task:

- We can define new concepts before we make tasks or observations where we use these concepts.
- We can solve many tasks under the same assumptions, observations and definitions.
- We do not have to present all assumption right away, we can include them as we need them.

Structured derivations generalize the constructs that we have described earlier. A structured task is a special case of a structured derivation, a derivation with only one task. A single assumption, declaration, fact and definition is each also a special cases of a structured derivation.

4.1 Definition

There are two new constructs here, a declaration and a definition. A definition has the general form

+ Define $c_1 \in A_1, \ldots, c_m \in A$

  {justification}

  definition condition

This defines the new constants $c_1 \in A_1, \ldots, c_m \in A_m$, which we can then use freely in the subsequent derivation. The justification should show, based on the assumptions and earlier observations, we can give these constants values that satisfy the definition condition. A defined constant can be simple, like a real number or an integer, but it can also be more complex, like a function.

We can use definitions also in structured tasks, in the same way as facts. They can be useful, e.g., when we need a shorter notation for some complicated concept. However, a definition will usually have a more general purpose, to introduce a new concept that is used in many places. Then it is natural that the definition is part of the general development of the theory, i.e. it is a step in a structured derivation.

Example 15. We define the sequence $a_0, a_1, a_2, \ldots$ as follows:

$$a_n = \frac{n}{2n+1}$$

when $n = 0, 1, 2, 3, \ldots$. Show that (A) $0 < a_n < \frac{1}{2}$ when $n \geq 1$, that (B) $a_{n+1} > a_n$ when $n \geq 0$ and (C) calculate $\lim_{n\to\infty} a_n$.

We solve this task by a general derivation. We identify the tasks by upper case letters, A, B and C.

+ Define $a : \mathbb{N} \to \mathbb{R}$

  {Function a describes a sequence, when we denote $a_i = a(i)$, $i = 0, 1, 2, \ldots$. This sequence is well-defined, since $2n + 1 > 0$ when $n = 0, 1, 2, \ldots$ }

  $a_n = \dfrac{n}{2n+1}$ when $n = 0, 1, 2, 3, \ldots$

A. Show that $0 < a_n < \dfrac{1}{2}$, when

\- $n \in \mathbb{N}$, $n \geq 1$

$\Vdash$ $0 < a_n < \dfrac{1}{2}$

$\equiv$ {definition a_n}

$$0 < \frac{n}{2n+1} < \frac{1}{2}$$

$\equiv$ {multiply both sides by $2n+1$, write the double inequality as a conjunction}

$$0 < n \wedge n < \frac{2n+1}{2}$$

$\equiv$ {simple}

$$0 < n \wedge 2n < 2n+1$$

$\equiv$ {according to the assumption $n \geq 1$, so the first proposition is true; the second proposition is always true}

$$T$$

$\square$

B. Show that $a_{n+1} > a_n$, when

\- $n \in \mathbb{N}$

$\Vdash$ $a_{n+1} > a_n$

$\equiv$ {definition a_n}

$$\frac{n+1}{2(n+1)+1} > \frac{n}{2n+1}$$

$\equiv$ {simple}

$$\frac{n+1}{2n+3} > \frac{n}{2n+1}$$

$\equiv$ {multiply by the expression $(2n+3)(2n+1)$, which is positive according to the assumption}

$$(2n+1)(n+1) > (2n+3)n$$

$\equiv$ {simple}

$$2n^2 + 3n + 1 > 2n^2 + 3n$$

$\equiv$ {subtract $2n^2 + 3n$ from both sides}

$$1 > 0$$

$\equiv$ {arithmetics}

T

$\square$

C. Define $\lim_{n\to\infty} a_n$

$\Vdash$ $\lim_{n\to\infty} a_n$

$=$ {according to the definition}

$\lim_{n\to\infty} \frac{n}{2n+1}$

$=$ {divide the numerator and the denominator by n }

$\lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{2n}{n}+\frac{1}{n}}$

$=$ {simple}

$\lim_{n\to\infty} \frac{1}{2+\frac{1}{n}}$

$=$ $\{\frac{1}{n} \to 0$ when $n \to \infty\}$

$\frac{1}{2}$

$\square$ $\lim_{n\to\infty} a_n = \frac{1}{2}$ ■

4.2 Modeling

Structured derivations are very useful when we want to model some situation and then ask questions about this model. We will solve the task we presented earlier about the train connection between Helsinki and Lappeenranta again, but now using a structured derivation. This means that we first construct a model that describes the task, and then we solve the questions related to the model.

It is often good to introduce the quantities that we use in a model explicitly. We do this with a *declaration,* of the form

$+$ $c_1 \in A_1, \ldots, c_m \in A$

This introduces new constants named $c_1, \ldots, c_m$ in the derivation, where the value of constant c_1 is an element of the set A_1, constant c_2 is an element of the set A_2 etc.

A declaration can be seen as a special case of a definition where we have omitted both the justification and the definition condition. This means that we only give the name and the value range of the constant (the value range must be non-empty). We can add further restrictions on the constant later on with assumptions.

Example 16. Since the year 1960 the travel time of the fastest train connection between Helsinki and Lappeenranta has decreased by 37%. Calculate by how may percent the average speed has increased. Assume that the length of the railroad has not changed.

We start by naming the constants that we use to describe the task.

+ $s \in \mathbb{R}$ – the distance between Helsinki and Lappeenranta

+ $t \in \mathbb{R}$ – the original travel time

+ $t' \in \mathbb{R}$ – the current travel time

We can write comments in a derivation after the "–" sign. The comments reminds us of what the quantities stand for in the original problem description.

We do not have to write the entire structured derivation as a single piece continuous text, we can add text between the steps as shown here. This helps us to explain to the reader how the solution proceeds, and explain the choices we make.

Next, we state the assumptions:

(a) $t' = 0.63 \cdot t$

(b) $s > 0, t > 0, t' > 0$

Notice assumption (b), which states that s, t, t' are all greater than zero. This follows from the definition of the task. Since we know that $s > 0$, the travel time cannot be zero.

We define the speeds in the usual way:

[1] Define $v \in \mathbb{R}$ – the original mean speed of the train

$\{v$ is well-defined, since $t > 0\}$

$$v = \frac{s}{t}$$

[2] Define $v' \in \mathbb{R}$ – the current mean speed of the train

$\{v'$ is well-defined, since $t' > 0\}$

$$v' = \frac{s}{t'}$$

We define the percentage of change in the speed as follows:

[3] Define $p \in \mathbb{R}$ – the percentage of change in train speed

$\{p$ is well-defined: since $s, t > 0$, then also $v > 0\}$

$$p = \frac{v' - v}{v}$$

We can now solve the task:

• p

$=$ {definition [3]}

$$\frac{v' - v}{v}$$

$=$ {definition [1] and [2]}

$$\frac{\frac{s}{t'} - \frac{s}{t}}{\frac{s}{t}}$$

$=$ {simplify}

$$\frac{\frac{s}{t'}}{\frac{s}{t}} - 1$$

$=$ {simplify the fractions}

$$\frac{s \cdot t}{s \cdot t'} - 1$$

$=$ {simplify, assumption}

$$\frac{1}{0,63} - 1$$

$\approx$ {calculate an approximate value}

0.59

$=$ {convert to percent: $x\% = \frac{x}{100}$}

59%

□

Answer: the mean speed has increased by 59%. ■

A structured derivation is a better way to present a mathematical argument than a structured task when the argument is long and there is a need to explain the different derivation steps and why we proceed in the way we do.

Chapter **5**

Derivations and Logic

A central feature of a structured derivation is the use of logical expressions and rules in mathematical proofs and derivations. Logic is of course an essential part of every proof, but it is often used in an informal way. Logical notation is used in an arbitrary and inconsistent manner. Standard logic notation is used systematically in structured derivations, and the rules of logic are used explicitly. This means that we can calculate with logical expressions in the same way as we calculate with standard arithmetic expressions.

5.1 Logical Connectives

We have collected the logical notations used in structured derivations into the table 5.1. The logical notations that we use in examples is fairly standard. Alternative notation can be, e.g., using "$\rightarrow$" for implication, and using "$\Leftrightarrow$" and "$\leftrightarrow$" for equivalence. Sometimes we see the notation "&" for conjunction and "|" for disjunction. The universal quantifier is sometimes denoted "Ax" and the existential quantifier sometimes "Ex".

We have already used logical notation in earlier examples quite freely. In fact, high

Table 5.1: Logical symbols

T	:	*truth*, a true proposition
F	:	*false*, a false proposition
$\neg p$	:	*negation*, proposition $\neg p$ is false when p is true, true when p is false
$p \wedge q$	:	*conjunction*, proposition is true when both p and q are true
$p \vee q$	:	*disjunction*, proposition is true when p or q (or both) are true
$p \Rightarrow q$	:	*implication*, if p is true, then q is also true
$p \equiv q$	:	*equivalence*, p and q are both true or both false
$(\forall x : p\,(x))$	:	*universal quantifier* $p(x)$ is true for every value of x
$(\exists x : p\,(x))$	:	*existential quantifier* , $p(x)$ is true for some value of x

school mathematics uses quite a bit of logic, both to present and to manipulate mathematical propositions. Our first example shows how logic is used when solving a second-degree equation.

Example 6. The task is to solve the second-degree equation $7x^2 - 6x = 0$. The following structured derivation solves the equation:

$\bullet$ Solve the equation $7x^2 - 6x = 0$

$\Vdash$ {equivalence is transitive}

$7x^2 - 6x = 0$

$\equiv$ {the distributive law: $a\,(b + c) = ab + ac$}

$x\,(7x - 6) = 0$

$\equiv$ {the zero-product property: $ab = 0 \equiv (a = 0 \vee b = 0)$}

$x = 0 \vee 7x - 6 = 0$

$\equiv$ {solve the equation on the RHS of the disjunction}

$x = 0 \vee x = \frac{6}{7}$

$\square$ $\blacksquare$

The logical proposition $7x^2 - 6x = 0$ is here modified (using steps that preserve the equivalence) into the expression $x = 0 \vee x = \frac{6}{7}$, which directly show which two values of the variable x satisfy the equation (the variable x satisfies the equation when its value is 0 or $\frac{6}{7}$). Each step is justified by a rule. For instance, in the first step we use the distributive law for multiplication. According to the rule, $a\,(b + c) = ab + ac$. We use this rule in the step

$$7x^2 - 6x = 0 \quad \equiv \quad x\,(7x - 6) = 0,$$

i.e. the rewritten equation is equivalent to the original equation.

It is worth noting that we give the solution to the second-degree equation as a disjunction. Thus we showed with the derivation above that the equation $7x^2 - 6x = 0$ is as true as the equation $x = 0 \vee x = \frac{6}{7}$. Thus an equation is a logical expression, which can be true for some values of the variable x and false for some other values of the variable x.

We can also use propositions expressed in natural language. The following example describes this.

Example 7. Show that k^2+k is an even number for every integer k. The following is a simple proof of the proposition:

$\bullet$	Show that $k^2 + k$ is an even number, when
-	k is an integer
$\Vdash$	the number $k^2 + k$ is even
$\equiv$	{the distributive law}
	the number $k(k+1)$ is even
$\equiv$	{a product is even if one of the factors is even}
	the number k is even $\vee$ the number $k+1$ is even
$\equiv$	{one of two consecutive integers is always even}
	T
$\square$	

$\blacksquare$

In the example we have used natural language in logical propositions, e.g. "$k^2 + k$ is an even number".

The task was to prove that the proposition "$k + k$ is an even number" is true. We express this in the following way

$$(k^2 + k \text{ is an even number}) \equiv T,$$

in other words, the proposition "$k^2 + k$ is an even number" is equivalent to the truth value T. It we want to prove the proposition using a derivation, then we write the propositions with equivalence relations. Since $p \Rightarrow T$ is always true, it is actually sufficient to show that the proposition $T \Rightarrow p$ is true.

The following example describes how we can use rules for logical expressions when we solve high school level problems.

Example 9. In a right triangle the length of the hypotenuse is 15 cm, and the perimeter is 36 cm. Find the lengths of the legs.

We use the Pythagorean theorem $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse. We can calculate the perimeter in the usual way as the sum of the side lengths, i.e. $a + b + c$.

$\bullet$	Calculate the length of the sides in a triangle, when
(a)	the triangle is a right triangle with the legs a and b and the hypotenuse c
(b)	$c = 15$ (cm)

(c) the perimeter of the triangle is 36 (cm)

$\Vdash$ T

$\equiv$ {the Pythagorean theorem, assumption (a)}

$a^2 + b^2 = c^2$

$\equiv$ {assumption (b) and (c)}

$a^2 + b^2 = 15^2 \wedge a + b + 15 = 36$

$\equiv$ {solve the second equation for b}

$a^2 + b^2 = 15^2 \wedge b = 21 - a$

$\equiv$ {substitute b from the second equation into the first equation, $15^2 = 225$}

$a^2 + (21 - a)^2 = 225 \wedge b = 21 - a$

$\equiv$ {calculate $(21 - a)^2$}

$a^2 + 441 - 42a + a^2 = 225 \wedge b = 21 - a$

$\equiv$ {simplify the first equation}

$2a^2 - 42a + 216 = 0 \wedge b = 21 - a$

$\equiv$ {solve the second-degree equation}

- $2a^2 - 42a + 216 = 0$

  $\equiv$ {the quadratic formula}

  $a = \frac{-(-42) \pm \sqrt{42^2 - 4 \cdot 2 \cdot 216}}{2 \cdot 2}$

  $\equiv$ {simplify}

  $a = \frac{42 \pm \sqrt{1764 - 1728}}{4}$

  $\equiv$ {simplify}

  $a = \frac{42 \pm 6}{4}$

  $\equiv$ {simplify}

  $a = 9 \vee a = 12$

  $\square$

$\ldots$ $(a = 9 \vee a = 12) \wedge b = 21 - a$

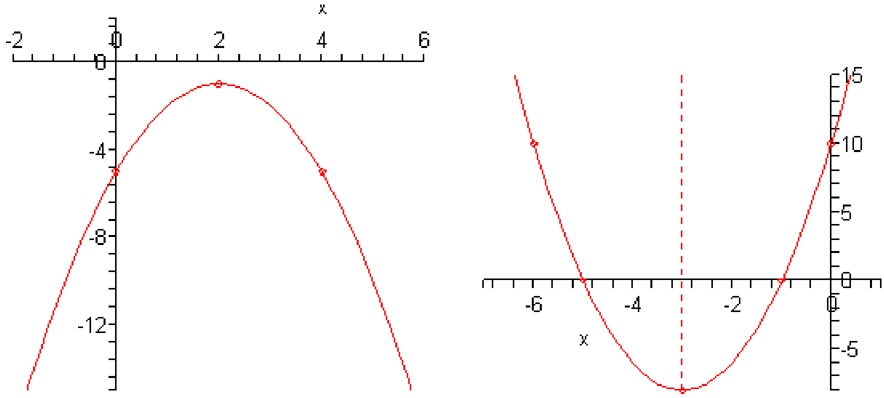
$\equiv$ {distributivity: $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$}

$(a = 9 \wedge b = 21 - a) \vee (a = 12 \wedge b = 21 - a)$

$\equiv$ {substitute the value of the variable a into the equation for b}

$(a = 9 \wedge b = 21 - 9) \vee (a = 12 \wedge b = 21 - 12)$

Figure 5.1: A parabola opening up and down



$\equiv$ {simplify}

$(a = 9 \wedge b = 12) \vee (a = 12 \wedge b = 9)$

□

The answer shows that the length of one leg is 9 cm the other is 12 cm. ■

We started the proof from the proposition T. Since we know that the Pythagorean theorem is true, we can use assumption (a) to show that

$$T \;\equiv\; a^2 + b^2 = c^2$$

Then we show that $a^2 + b^2 = c^2$ is equivalent to the proposition $(a = 9 \wedge b = 12) \vee (a = 12 \wedge b = 9)$. Since T is true, the proposition that we get as a result must be true, and we have solved the task.

5.2 Quantifiers in High School Mathematics

The following task is an example of a more challenging derivation, where we show how to solve a mathematical problem using logical notation, in this case quantifier notation.

Example 8. For which values of the constant a is the function $f : \mathbb{R} \to \mathbb{R}$ always negative, when $f(x) = -x^2 + ax + a - 3$ for every x?

In this case, solving the main task require that we solve two subtasks (calculate the discriminant D_f and calculate its zeros). We refer to the curves in the figure 5.1, which are parabolas opening up and down:

We solve the task using a structured derivation in the following way.

- Calculate for which values of the constant a the function f is always negative, when

- $f(x) = -x^2 + ax + a - 3$ for every $x \in \mathbb{R}$

$\Vdash$ $\left(\forall x : -x^2 + ax + a - 3 < 0\right)$

$\equiv$ {the graph of function f is a parabola opening down when the coefficient of the second-degree term is negative; a graph is always negative, if it does not have zeros (the left-hand figure)}

$\left(\forall x : -x^2 + ax + a - 3 \neq 0\right)$

$\equiv$ {a second-degree equation does not have zeros, if the discriminant D_f is less than zero}

$D_f < 0$

$\equiv$ {substitute the value of the discriminant D_f}

- Calculate the value of the discriminant D_f

  $\Vdash$ D_f

  $=$ { the discriminant of $Ax^2+Bx+C = 0$ is given by the formula B^2-4AC}

  $a^2 - 4(-1)(a-3)$

  $=$ {simplify}

  $a^2 + 4a - 12$

  $\square$

$\ldots$ $a^2 + 4a - 12 < 0$

$\equiv$ {the graph of the function defined by the expression $a^2 + 4a - 12$ is a parabola opening up when the coefficient of the second-degree term is positive, thus the graph is negative between the zeros (the right-hand figure)}

- Calculate the zeros of the polynomial $a^2 + 4a - 12$

  $\Vdash$ $a^2 + 4a - 12 = 0$

  $\equiv$ {the quadratic formula}

  $a = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1}$

  $\equiv$ {solve}

  $a = 2 \vee a = -6$

  $\square$

$\ldots$ $-6 < a < 2$

$\square$

We have thus proved that

$$\left(\forall x : -x^2 + ax + a - 3 < 0\right) \equiv -6 < a < 2.$$

In other words, the function f is negative, if and only if $-6 < a < 2$. ■

The main derivation and one of the nested derivations use equivalence between the logical proposition, while in the first nested derivation uses equality between arithmetic expressions.

We can also use extra material, such as figures, tables or other aid in structured derivations. In the previous example there was two figures that we referred to in the proof. Additional material is presented outside the proof, but it can be referred to in the derivation. We also use the properties of parabolas in the proof.

We need to prove that the relation between the proposition is equivalence. If we only proved the implication to the right

$$(\forall x : -x^2 + ax + a - 3 < 0) \Rightarrow -6 < a < 2,$$

the conditions for the constant a might be too weak, which would mean that we could get extra values that do not satisfy the original conditions. If on the other hand, we only showed that

$$\big(\forall x : -x^2 + ax + a - 3 < 0\big) \Leftarrow -6 < a < 2,$$

it is possible that we would not find every value of the constant a that satisfies the original condition. We use equivalence to show that the condition is satisfied for the calculated values of the constant a and only for those values.

5.3 Exact Proofs

The justifications for the steps have been rather free until now. If we want to make the proofs logically more precise, we need to mention which rule we have used in each step. We must also mention how we have used the rule and justify why we are allowed to use the rule in this context.

Let us study the distributive law, which is a typical rule from algebra:

$$x(y + z) = xy + xz.$$

This is true for every arithmetic expression x,y and z. Here x,y and z are so-called metavariables (or syntactic variables), which we replace by concrete expressions when we use the rule.

We can write the exact justification of a proof step in the form

$$\{name : rule, \text{ where } substitution, \text{ condition } assumptions \text{ holds}\}$$

Here

- the name of the rule is mentioned (*name*),

- the rule itself is written explicitly if necessary (*rule*),
- we describe how we use the rule (i.e. which values are substituted for the syntactic variable) (*substitutions*) and
- we state that the *assumptions* for using the rule are satisfied.

We can the write a more exact justification for a step where we use the distributive law in the form:

$$
\begin{array}{ll}
 & (a-b)\,(a+b) \\
= & \{\text{the distributive law for addition: } x\,(y+z) = xy + xz, \text{ where } x := a-b, \\
 & y := a,\ z := b, \text{ condition } a-b,\ a \text{ and } b \text{ are arithmetic expressions holds}\} \\
 & (a-b)\,a + (a-b)\,b
\end{array}
$$

Here $x := a-b$, $y := a$ and $z := b$ are substitutions used in the rule. Thus we apply the distributive law so that we choose $a-b$ as the variable x, a as the variable y and b as the variable z. We also state that we can use the rule because every expression is arithmetic. After substituting we get a special case of the distributive law:

$$(a-b)(a+b) = (a-b)a + (a-b)b$$

We can write the rule directly into the justification, as we have done earlier, or then we only write the name of the rule. We often omit the substitution, if it is clear from the proof step which substitution we have used. In the same way we can omit the conditions, if the context make them clear. In this case we can provide a very short justifications for the proof step,

$$
\begin{array}{ll}
 & (a-b)(a+b) \\
= & \{\text{the distributive law for addition}\} \\
 & (a-b)a + (a-b)b
\end{array}
$$

The reader then has to add the necessary details, i.e. what the rule actually says and which substitution we have used. Furthermore we need to check that the assumptions for using the rule hold.

We will now prove the formula for the difference of two squares again, but this time we only use axioms for real numbers. We use nested derivations in the proof to give it the same structure as the earlier proof, i.e. the main steps are the same.

Example 17.

•   Show that $(a-b)(a+b) = a^2 - b^2$, using only axioms for real numbers

$\Vdash$ {transitivity}

$(a-b)(a+b)$

$=$ {prove using axioms}

$\ldots$ $a^2 + (-ba) + ba + \left(-b^2\right)$

$=$ {prove using axioms}

$\ldots$ $a^2 - b^2$

$\square$

The nested derivations are hidden in this proof, so that we can see the overall structure of the proof. There are two nested derivations in the proof. The first one is the following:

•   Prove that $(a-b)(a+b) = a^2 - b^2$ using axioms

$\Vdash$ $(a-b)(a+b)$

$=$ {the distributive law for addition: $x(y+z) = xy + xz$, where $x := a - b$, $y := a$, $z := b$}

$(a-b)a + (a-b)b$

$=$ {the commutative law: $xy = yx$, where $x := a - b$, $y := a$}

$a(a-b) + (a-b)b$

$=$ {the commutative law: $xy = yx$, where $x := a - b$, $y := b$}

$a(a-b) + b(a-b)$

$=$ {the definition of subtraction: $x - y = x + (-y)$, where $x := a$, $y := b$}

$a(a + (-b)) + b(a + (-b))$

$=$ {the distributive law of addition: $x(y+z) = xy + xz$, where $x := a$, $y := a$, $z := -b$}

$aa + a(-b) + b(a + (-b))$

$=$ {the distributive law of addition: $x(y+z) = xy + xz$, where $x := b$, $y := a$, $z := -b$}

$aa + a(-b) + ba + b(-b)$

$=$ {the commutative law: $xy = yx$, where $x := a$, $y := -b$}

$aa + (-b)\,a + ba + b\,(-b)$

$=$ {multiplication by additive inverse: $(-x)\,y = -\,(xy)$, where $x := b$, $y := a$ and $y := b$}

$aa + (-ba) + ba + (-bb)$

$=$ {the definition of the power x^2: $x^2 = xx$, where $x := a$ och $x := b$}

$a^2 + (-ba) + ba + \left(-b^2\right)$

□

The nested derivation of the second step is the following:

• Prove that $a^2 + (-ba) + ba + \left(-b^2\right) = a^2 - b^2$ using axioms

$\Vdash$ $a^2 + (-ba) + ba + \left(-b^2\right)$

$=$ {the sum of additive inverses: $-x + x = 0$, where $x := ba$}

$a^2 + 0 + \left(-b^2\right)$

$=$ {add zero: $x + 0 = x$, where $x := a^2$}

$a^2 + \left(-b^2\right)$

$=$ {the definition of subtraction: $x - y = x + (-y)$, where $x := a^2$, $y := b^2$}

$a^2 - b^2$

□ ■

This more exact proof is much longer because it involves only axioms for real numbers. When we hide the nested derivations, we are left with the original proof. This shows how we can argue at different levels of detail with structured derivations. Depending on whom the proof is aimed at, we can present an exact or a crude version by showing or hiding the nested derivations.

Chapter **6**

Syntax of Structured Derivations

A structured derivation has the following general form

***derivation*:**

derivation step$_1$

derivation step$_2$

$\vdots$

derivation step$_n$

In other words, a structured derivation is a sequence of consecutive derivation steps. An individual derivation step is either an assumption, a declaration, a fact, a definition or a (structured) task:

***derivation* step:**

assumption | *declaration* | *fact* | *definition* | *task*

We have described the general format for the first four constructs earlier. The general form of a task is as follows:

***task*:**

•	*question*
-	*assumption*
⋮	
-	*assumption*
+	*declaration*
	justification
	proposition
⋮	
+	*declaration*
	justification
	proposition
$\Vdash$	*justification*
	expression
rel	*justifictation*
	expression
⋮	
rel	*justification*
	expression
□	*answer*

***justification*:**

{*explanation*}

task

⋮

task

A *definition* above consists of a declaration part, a justification and a proposition (the definition condition). A *fact* is a kind of definition, where we do not introduce any new constants. In that case, we omit the declaration part of the definition. A *declaration* will again omit the justification and the proposition part of the definition. A structured calculation is a special case of a structured task, where there are no question, no assumptions, no declarations, no facts or definitions, and no answer. We then also omit the sign $\Vdash$ and the following justification from a calculation.

Notice that a *task* is defined in terms of *justifications* and that a *justification* is defined in terms of *tasks*. The task is thus defined in *recursive* manner.

The form of structured derivation presented above leaves a number of concrete details open. This is intentional, the idea is that these are determined by the underlying mathematical theories that used in the derivation. To fix the syntax of structured derivations completely, we should also define how to write the following parts of a derivation:

- *assumption* — logical proposition
- *proposition* — logical proposition
- *explanation* — an explanation for why a derivation step is justified
- *expressions* — the expressions permitted by the theory in use
- *rel* — the relation between the calculation steps
- *question* – what is the task
- *answer* – the answer to the task
- *declaration* – new names of constants that we introduce

We leave these categories undefined, however, so that we can use the presentation format that we want, depending on which level of education we use the derivation, which branch of mathematics we apply the method on and with what precision we express the logic we use.

Chapter 7

Additional Information

Structured derivations are described in more broadly and in more detail in the book

> *Teaching Mathematics in the Digital Age with Structured Derivations* (Ralph-Johan Back, Four Ferries Publishing, 2016).

A shorter version of this book is

> *Structured Derivations: Teaching Mathematical Reasoning in High School* (Ralph-Johan Back, Four Ferries Publishing, 2015),

which focuses on the use of the method on secondary level education. Both books are available from e.g. the Amazon book store.

To see how structured derivations are used in practice in mathematics education, you can check out the *eMath* digital book. This covers all the compulsory courses of the standard high school mathematics curriculum in Finland (grades 10 - 12).

> *eMath 1 - 10* (Ralph-Johan Back, Stefan Asikainen, Matti Hutri, Joonatan Jalonen, Antti Lempinen, Marie Linden-Slotte, Saara Mäkinen, Petri Sallasmaa, Petri Salmela. Four Ferries Publishing 2016).

The book series is available on iPad tablets from the AppStore and for Android tablets from Google Play.

The *4f Studio* application supports writing structured derivations on computers. The system includes an electronic notebook, where one can write mathematical text, create and edit structured derivations and make graphs of functions, geometric figures and mathematical tables. *The Four Ferries* web pages (*www.fourferries.fi*) give more information about 4f Studio. There you can also find tutorials about structured derivations and digital mathematics education as well as educational videos and information about research and development of structured derivations and practical experiences of the use of the method on different levels of education.

Introduction to Structured Derivations

Structured derivations is a method that supports the construction, presentation and understanding of mathematical arguments. The method works equally well for mathematical proofs and algebraic and arithmetic calculations as for geometric constructions and general problem solving, and it is useful whenever the presentation of a solution requires several consecutive steps. It has been used at different levels of mathematics, from lower secondary school to university level and research. The method is based on a fixed form to present mathematical arguments and the use of simple logical notation in the arguments. The fixed form makes it easier to to understand proofs and calculations and to check that they are correct. The aim of this guide is to show how structured derivations can be used in upper secondary school level mathematics education. The method is described with examples that step by step introduce new features and concepts.

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