PROMOTING STUDENTS' JUSTIFICATION SKILLS USING STRUCTURED DERIVATIONS

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Being able to explain the process of solving a mathematical problem is essential to learning mathematics. Unfortunately, students are not used to justifying their solutions as emphasis in the classroom is usually put on the final answer. In this paper, we describe how students can become used to explicate their thinking while solving a problem or writing a proof in a structured and standard format using structured derivations. We also present the results from an analysis of upper secondary school students' argumentation skills from using this approach in a course on logic and number theory. Our findings suggest that the structured derivations format is appreciated by the students and can help promote their justification skills.

BACKGROUND

"Mathematics is not just about identifying the truth but also about proving that this is the case" (Almeida, 1995, p. 171). Learning to argue about mathematical ideas and justifying solutions is fundamental to truly understanding mathematics and learning to think mathematically.

The National Council of Teaching Mathematics (NCTM) issues recommendations for school mathematics at different levels. In the current documents (NCTM, 2008), communication, argumentation and justification skills are recognized as central to the learning of mathematics at all levels.

According to Sfard (Sfard, 2001), thinking can be seen as a special case of intrapersonal communication: " [o]ur thinking is clearly a dialogical endeavor where we inform ourselves, we argue, we ask questions, and we wait for our own response […] becoming a participant in mathematical discourse is tantamount to learning to *think* in a mathematical way" (p.5). Although it is important to be able to communicate mathematical ideas orally, documenting the thinking in writing can be even more efficient for developing understanding (Albert, 2000).

Justifications are not only important to the student, but also to the teacher, as the explanations (not the final answer) make it possible for the teacher to study the growth of mathematical understanding (Pirie & Kieren, 1992). Using arguments such as "Because my teacher said so" or "I can see it" is insufficient to reveal their reasoning (Dreyfus, 1999). A brief answer such as "26/65=2/5" does not tell the reader anything about the student's understanding. What if he or she has "seen" that this is the result after simply removing the number six (6)?

Nevertheless, quick and correct answers are often valued more in the classroom than the thinking that resulted in those answers. It is common for students to be required to justify their solution and explain their thinking only when they have made an error – the need to justify correctly solved problems is usually de-emphasized (Glass & Maher, 2004). As a result, students rarely provide explanations in mathematics class and are not used to justify their answers (Cai et al., 1996). Consequently, the reasoning that drives the solution forward remains implicit (Dreyfus, 1999; Leron, 1983).

In this paper, we will present an approach for doing mathematics carefully, which aids students in documenting their solutions and their thinking process. We will also present the results from the analysis of students' justifications from a course using this approach. The aim is to investigate the following questions:

- How does the use of structured derivations affect students' justifications?
- What advantages and drawbacks do students experience when using structured derivations?

STRUCTURED DERIVATIONS

Structured derivations (Back et al., 1998; Back & von Wright, 1999; Back et al., 2008) is a further development of Dijkstra's calculational proof style, where Back and von Wright have added a mechanism for doing subderivations and for handling assumptions in proofs. With this extension, structured derivations can be seen as an alternative notation for Gentzen like proofs.

In the following, we illustrate the format by briefly discussing an example where we want to prove that $x^2 > x$ when $x > 1$.

• Prove that $x^2 > x$, when

- $x > 1$

||- $x^2 > x$

≡ { Add $-x$ to both sides }

  $x^2 - x > 0$

≡ { Factorize }

  $x(x - 1) > 0$

≡ { Both x and x-1 are positive according to assumption. Therefore their product is also positive. }

  T

The derivation starts with a description of the problem ("Prove that $x^2 > x$"), followed by a list of assumptions (here we have only one: $x > 1$). The turnstile (||-) indicates the beginning of the derivation and is followed by the start term ($x^2 > x$). In this example, the solution is reached by reducing the original term step by step. Each step in the derivation consists of two terms, a relation and an explicit justification for why the first term is transformed to the second one. Justifications are written inside curly brackets.

Another key feature of this format is the possibility to present derivations at different levels of detail using subderivations, but as these are not the focus of this paper, we have chosen not to present them here. For information on subderivations and a more detailed introduction to the format, please see the book and articles by Back et al.

Why Use in Education?

As each step in the solution is justified, the final product contains a documentation of the thinking that the student was engaged in while completing the derivation, as opposed to the implicit reasoning mentioned by Dreyfus (1999) and Leron (1983). The explicated thinking facilitates reading and debugging both for students and teachers.

Moreover, the defined format gives students a standardized model for how solutions and proofs are to be written. This can aid in removing the confusion that has commonly been the result of teachers and books presenting different formats for the same thing (Dreyfus, 1999). A clear and familiar format has the potential to function as mental support, giving students belief in their own skills to solve the problem. As solutions and proofs look the same way using structured derivations, the traditional "fear" of proof might be eased. Furthermore, the use of subderivations renders the format suitable for new types of assignments and self-study material, as examples can be made self-explanatory at different detail levels.

STUDY SETTINGS

The data were collected during an elective advanced mathematics course on logic and number theory (about 30 hours) at two upper secondary schools in Turku, Finland during fall 2007. Twenty two (22) students participated in the course (32 % girls, 68 % boys). The students were on their final study year.

For this study, we have used a pre course survey including a pretest, three course exams and a mid and post course survey. The pretest included five exercises, which students were to solve. They were also asked explicitly to justify their results. The surveys included both multiple choice questions and open-ended questions for students to express their opinions in their own words.

For each course exam, we have manually gone through and analyzed three assignment solutions per student, giving us a total of 198 analyzed solutions (22 students * 3 exams * 3 solutions). In the analysis, we focused on two things: the types of justification related errors (JRE) and the frequency of these.

RESULTS AND DISCUSSION

Justification related errors in the exams

The analysis revealed the following three JRE types:

- *Missing justification*. A justification between two terms in the derivation is missing.

- *Insufficient or incorrect justification*. E.g. using the wrong name of a rule or not being precise enough, for instance, writing "logic" as the justification, when a more detailed explanation would have been needed.
- *Errors related to the use of mathematical language*. Characterized by the student not being familiar with the mathematical terminology. For instance, one student wrote "solve the equation" when actually multiplying two binomials or simplifying an inequality.

The pre course survey indicated that the students had quite varied justification skills. Over half of the students disagreed with the statement "I usually justify my solutions carefully" and an analysis of the pretests showed that many students did do quite poorly on the justification part, especially for the two most difficult exercises (over 50 % of the students gave an incorrect or no explanation). Also, the nature of the justifications was rather mixed: whereas some gave detailed explanations, some only wrote a couple of words giving an indication of what they had done.

The exam assignments included surprisingly few JREs taking into account the skills exhibited by students in the pretest. The overall frequency of JREs stayed rather constant throughout the course: a JRE was found in 15-20 % of the 66 assignments analyzed for each exam. Most students who made a JRE of a specific type, made only one such error in the nine assignments. Note that this is one erroneous justification comment throughout all three exams. Only six students made more than one JRE of a specific type.

Missing justifications were the most common JRE in the second exam (11 % of students), whereas students did mainly insufficient/ incorrect justifications in the first and third exam (9-12 %). Errors related to mathematical language stayed fairly constant in all exams (3-6 %).

The low number of missing justifications in the first exam is understandable given the character of the assignments (short, familiar topics). In the second exam, new topics had been introduced, resulting in a larger number of missing justifications. This however decreased in the third exam, suggesting that students had got used to always justifying each step. The slightly increased number of insufficient/ incorrect justifications in the third exam can be explained by the third exam being the most difficult one. The main point here is to note that the overall frequency of JREs was low.

Survey results

The mid and post course surveys revealed students' perceived benefits and drawbacks of using structured derivations. Our analysis showed that 77 % of the students stated that the solutions were much clearer than before. Further another 77 % suggested an increased understanding for doing mathematics.[1]

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[1] The quotations have been freely translated from Swedish by one of the authors.

> "At first I found it completely unnecessary to write this way, but now I think it is a very good way, because now I understand exactly how all assignments are done."

> "I actually liked this course (rare when it comes to mathematics), structured derivations made everything much clearer. Earlier, I basically just wrote something except real justifications. Sometimes I haven't known what I've been doing."

The main drawbacks, according to the students, were that the format made solutions longer (32 % of students) and more time consuming (55 % of students). This is understandable, as the explicit justifications do increase the length of the solutions and also take some time to write down. The justifications, however, were considered a source of increasing understanding, thus the time consumption might be regarded something positive after all. In fact, we believe it is a large benefit, as it helps promote quality instead of quantity.

The students also noted that structured derivations required more thinking. Moreover, they recognized that the format helped them make fewer errors partly because they had to let it take time to write down the solutions.

> "In this course the calculations become more careful since you take the time to think every step through."

> "[Using the traditional format, you] can more easily make mistakes when you calculate so fast."

Another interesting finding was that students seemed to believe that justifications were not part of the solutions when doing mathematics in the traditional format. Describing the traditional way they do mathematics, they e.g. noted:

> "You don't have to explain what you do!… It's enough to get a reasonable answer."

> "You lack explanations for why you do things the way you do."

A final remarkable observation was the lack of completely negative comments. Comments starting out in a negative tone ("It takes much time", "I don't like all the writing"), all ended up positive ("… but I understand what I do better", "…but I make fewer errors"). In our opinion, this is a promising finding.

CONCLUDING REMARKS

The format and results presented in this paper, suggest that it is possible to get students to start justifying their solutions better. If you want to do something carefully, it will take some time and effort. "Quality before quantity" is something that, in our opinion, should be emphasized also in mathematics education.

The focus on also explaining solutions raises a new challenge - how do we get students to choose an appropriate level of detail for their justifications. While talented students may feel comfortable using "simplify" as a justification, this might not be sufficient for weaker students. A certain level of detail thus needs to be enforced at least at the beginning of a new topic, in order to ensure that students truly are learning the topic at hand.

Another question raised that merits further investigations is what type of justification should be preferred (name of a mathematical rule, natural language description of the process, i.e. what is done in the step)? The impact of the type of justification ("simplify" compared to a longer description) on the quality/ correctness of a solution also deserves attention.

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