AN EMPIRICAL EVALUATION OF STRUCTURED DERIVATIONS IN HIGH SCHOOL MATHEMATICS

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Structured derivations is a new method for presenting mathematical proofs and derivations. It is based on a systematic and standardized way of describing mathematical arguments, and uses basic logic to structure the derivation and to justify the derivation steps. We have studied the use of structured derivations in high school in two successive controlled studies. The results indicate that using structured derivations gives a marked performance improvement over traditional teaching methods. We describe the structured derivations method, the set up of our study and our main results.

BACKGROUND

One of the cornerstones of mathematical reasoning is mathematical proof. However, proofs are considered difficult and consequently today's high school curricula typically mention proof only in connection with geometry. Strong arguments have been presented in favor of more training in rigorous reasoning (Hanna & Jahnke 1993, Hoyles 1997). Mathematical proofs are based on logic and logical notation, but using logic in proofs is usually not taught systematically in high schools today. Proofs in high school are therefore informal and not uniform. Where logic is taught, it is seen as a separate object of study, rather than as a tool to be used when solving mathematical problems.

Writing solutions to mathematics problems in an unstructured and informal way makes it hard for students to know when a problem has been acceptably solved. It also makes it difficult to look at solutions afterwards, study them and discuss them. Only in connection with a few specific problem areas (typically algebraic simplification and equation solving) is a more uniform format for writing solutions at hand. However, even then it is often not clear to students, e.g., why deriving 0=1 from an equation means that the equation has no solutions.

Structured derivations provide an alternative approach to teaching mathematics, based on systematic proofs and derivation and the explicit use of logical notation and logical inference rules. Structured derivations have been developed by Back and von Wright (Back et al, 1998; Back & von Wright, 1999; Back & von Wright, 2006; Back et al, 2008), first as a way for presenting proofs in programming logic, and later adapted to provide a practical approach to presenting proofs and derivations in high school mathematics. Structured derivations are a further development of the *calculational proof method* originally developed E.W. Dijkstra and his colleagues (see Dijkstra, 2002, for a summary and motivation). Structured derivations add a mechanism for doing subderivations and for handling assumptions in proofs to calculational proofs. Structured derivations can be seen as a combination of Dijkstra like calculational proofs and Gentzen like backward chaining proofs.

We have been experimenting with using structured derivations for teaching mathematics in high school, with very encouraging results. It seems that the standardized format provided by structured derivations helps the students in constructing a proof and in checking that their proof is correct, without being too formal and/or intimidating to be useful in practice.

We start below with a short overview of structured derivations, before we proceed to describe two large empirical studies that we have carried out to evaluate the use of our approach in teaching mathematics in high school.

STRUCTURED DERIVATIONS BY EXAMPLE

We illustrate structured derivations with a simple example: solve the equation $(x-1)(x^2+1)=0$. The solution is as follows:

- $(x-1)(x^2+1)=0$

$\equiv$ {zero product rule: $ab=0 \equiv a=0 \vee b=0$}

$x-1=0 \vee x^2+1=0$

$\equiv$ { add 1 to both sides of left disjunct}

$x=1 \vee x^2+1=0$

$\equiv$ {add -1 to both sides in right disjunct }

$x=1 \vee x^2=-1$

$\equiv$ {a square is never negative}

$x=1 \vee$ False

$\equiv$ {disjunction rule}

$x=1$

The original equation is transformed in a sequence of equivalence preserving steps to the solution “x=1”. Each step in the derivation consists of two terms, a relation and an explicit justification for why the first term is related to the second one in the indicated way. In this case, the terms are Boolean formulas, and the relation is equivalence between the terms.

This example does not show a number of important features in structured derivations, such as the possibility to present derivations at different levels of detail using subderivations, and the use of assumptions in proofs. These are not the focus of this paper, so we have chosen not to present them here. For information on subderivations and a more detailed introduction to the format, please see the articles by Back et al.

It is important that each step in the solution is justified. The final product will then contain a documentation of the thinking that the student was engaged in while completing the derivation, as opposed to the implicit reasoning mentioned by Dreyfus (1999) and Leron (1983). The explicated thinking facilitates reading and debugging both for students and teachers. It also leaves a more explicit documentation of the teacher's explanation of an example, making it easier for students to catch up later on issues they did not understand during the lectures.

Moreover, the defined format gives students a standardized model for how solutions and proofs are to be written. This can aid in removing the confusion that may result from teachers and books presenting different formats for the same thing (Dreyfus, 1999). A clear and familiar format also has the potential to function as mental support, giving students belief in their own skills to solve the problem, and the satisfaction of being able to check for themselves that they have indeed produced a correct solution. The use of subderivations renders the format suitable for new types of assignments and self-study material, as examples can be made self-explanatory at different levels of detail.

EMPIRICAL STUDY

Our purpose was to test whether teaching mathematics using structured derivations in high school (upper secondary education) would improve the students learning, as compared to teaching mathematics in the traditional way. High schools in Finland are 3-4 years and the students are 16 – 19 years old. Mathematics is taught at two levels, standard and advanced. Mathematics at the advanced level is in practice a pre-requisite for studying Science, Engineering, Medicine and Business Administration at University level. The advanced level is therefore quite popular, and is taken on average by 40 % of the students. There are altogether 10 compulsory mathematics courses on the advanced level, as well as some optional courses. High school ends with a national matriculation exam in mathematics, which is taken by almost all advanced level students.

We carried out our empirical studies at Kupittaa High School in Turku, Finland. This school offers extra courses in IT at high school level (programming and telecommunication courses). We carried out two 3-year empirical studies at this high school, the first 2001 – 2004, and the second 2002 – 2005.

Both studies were organized in the same way. The students starting high school were divided into three groups. First there was a test group consisting of those students who wanted to take some extra courses in IT. The remaining students were divided into two groups, a control group that was chosen so that its starting situation would be as similar as possible to the test group, and a third group consisting of all the remaining students. The third group did not participate in the study. The students had the final say on which groups to join, so it was not possible to make the test and the control groups exactly similar. The test and control groups were of approximately same size.

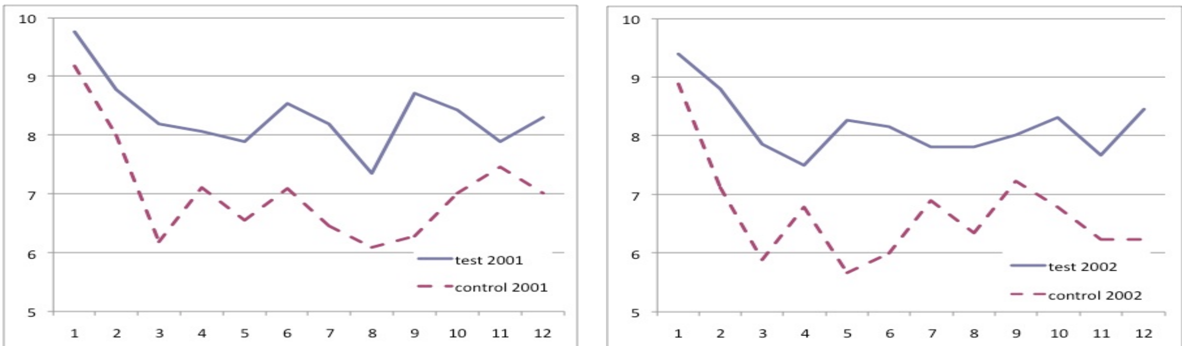
Groups had lectures and exams at exactly the same time, they followed the same curriculum, and they had exactly the same exams. The test group was taught all mathematics courses using structured derivations, while the control group was taught in the traditional way. The exception was the course in Geometry, which was taught in the traditional way also in the test group. The test group and the control group had different teachers.

The study measured the average performance of the test and the control groups on all ten mathematics courses, as well as on the final matriculation exam.

MAIN RESULTS

Graph 1 shows the performance of the students on the individual math courses, (a) shows the 2001 – 2004 study and (b) the 2002 – 2005 study. Courses are graded on a scale from 4 – 10, where 4 is not passed, 5 is the lowest grade (barely pass) and 10 is the best grade (excellent). The results of the test groups are shown as a solid (blue) line, while the results of the control groups are shown as a dashed (red) line.

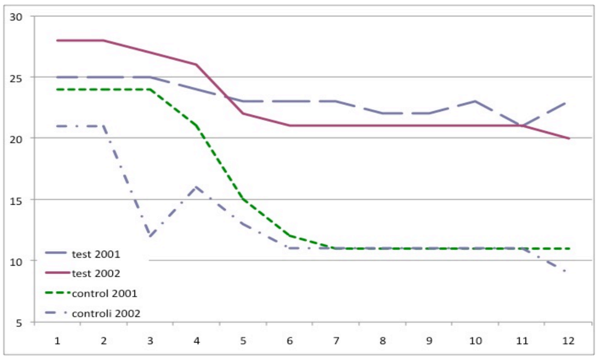
The diagram shows first the average grade in mathematics for each group when entering high school (x=1). Then the average grade for each group in each of the ten compulsory mathematics courses is shown (x= 2, 3,..., 11). Lastly, the average for each group in the national matriculation exam is shown (x = 12). The data for the two groups show in each study only those students who completed all 10 courses and took the final matriculation exam.

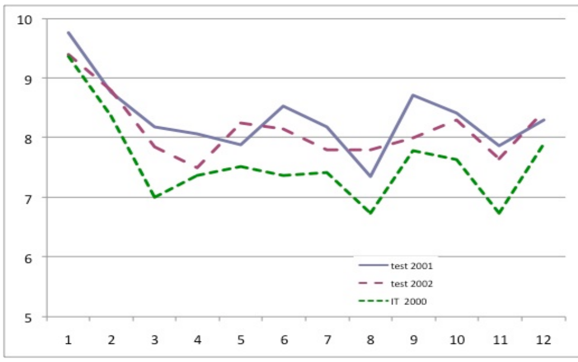


Graph 1: (a) Study 2001 – 2004 (b) Study 2002 – 2005

Both studies show that the test group performs markedly better than the control group in all courses, as well as in the final matriculation exam. This can be partly explained by the initial difference in the entrance scores for the two groups. However, the test group performance is much better than what one would expect from the initial difference in entrance grades alone. This is also supported by a more detailed analysis of the data.

The strength of the test group in both studies is further emphasized by the difference in attendance in the two groups (Graph 2 (a)). Each study followed the students throughout their three years at high school. The students have always an option to drop out of the group, either by moving from advanced level mathematics to standard level, or by performing badly in exams so that they need to retake courses and thus cannot anymore follow the same program as the rest of the test or control group.





Graph 2: (a) Attendants in both studies (b) Comparing IT-groups 2000-2003, 2001-2004 and 2002-2005

The graph shows that there is almost no dropout in the test group. The situation is quite different for the control group. The dropout rate is much higher, less than half of the students in the control group actually finished and took the matriculation exam in due time.

DISCUSSION

The main difference between the test group and the control group is the method of teaching: the test group uses structured derivations and the control group uses traditional teaching methods. But there are, of course, also other differences that could explain the results: the test groups have a somewhat higher average entry grade in mathematics than the control groups, the groups are taught by different teachers, and the students in the test group are there because they preferred to take IT related courses to some other courses. We can check whether these other differences can explain the results, by comparing the results of the same teacher teaching the group of students interested in IT that enrolled one year earlier, in 2000, and wrote their matriculation exam in 2003 (Graph 2 (b)). The selection criteria are thus the same for this group and for the two test groups (2001-2004 and 2002-2005). These two groups also happen to have exactly the same average entry grades in mathematics.

We see that the groups using structured derivations still outperform the group using traditional teaching methods. The difference is not as great as in the earlier comparisons, but it is still quite noticeable. We interpret this as showing that part of the difference between the test and the control groups in the earlier experiments can be attributed to the difference between teachers and entry grades, but not all. A marked difference in favor of the test group remains, indicating that the use of structured derivations really does improve mathematics learning for high school students.

We can also statistically compare the test group of 2002 to the IT group of 2000, because these two groups have the same average entry grades, same selection criteria were used, and the teacher was the same in both groups. A two sample t-test shows that the differences between the course averages in the two groups is statistically significant in 7 cases out of 10 ($0.01 < p < 0.1$, depending on the course). The difference in the matriculation exam is also statistically significant ($p < 0.1$). Two courses where no statistically significant difference was found were Geometry (which was taught in a traditional way also in the test group) and Integrals. A characteristic of the latter course is that it uses a calculational style of reasoning which is not that far away from the structured derivations method.

CONCLUDING REMARKS

The results seem to validate our hypothesis that the use of structured derivations does indeed improve the mathematics performance of high school students. The structured derivations approach to teaching mathematics seems very promising, with a potential for achieving marked improvement in learning results in high school.

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